

# QUANTIZATION OF CLASSICAL CURVES

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**ABSTRACT.** We discuss the relation between quantum curves (defined as solutions of equation  $[P, Q] = \hbar$ , where  $P, Q$  are ordinary differential operators) and classical curves. We illustrate this relation for the case of quantum curve that corresponds to the  $(p, q)$ -minimal model coupled to 2D gravity.

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It was shown by Krichever that a solution of equation  $[P, Q] = 0$ , where  $P, Q$  are ordinary differential operators can be obtained from two meromorphic functions on an algebraic curve  $X$  if we assume that these functions obey certain conditions. He proved similar results for ordinary differential operators with matrix coefficients and for finite difference operators.

These results led to the idea that a quantum curve can be identified with a solution of the equation  $[P, Q] = \hbar$  where  $P, Q$  are ordinary differential operators [7, 8, 1].

In Ref. [8] it was proven that under certain conditions the moduli space of solutions to the equation  $[P, Q] = \hbar$  can be identified with the moduli space of solutions to the equation  $[P, Q] = 0$ . This means that there exists a quantization procedure that is unique in some sense. In Ref. [5] this result was generalized to the matrix differential operators. More detailed consideration as well as the proof of a similar result for difference operators was given in Ref. [10]. The proofs in these papers are based on the techniques of Sato Grassmannian.

In the present paper we will give an explanation of the quantization procedure for commuting pairs of differential operators, describing the algorithmic part of Ref. [10]. We will illustrate this procedure on some examples. We will discuss shortly the relation to Eynard-Orantin topological recursion [2], to Gukov-Sułkowski quantization, [3] and to the theory of quantum curves developed in Ref. [1].

Let us consider the algebra  $\mathcal{D}$  of ordinary differential operators  $\sum a_k(x)D^k$  where  $D = \frac{d}{dx}$  and the coefficients  $a_k(x)$  are formal power series with respect to  $x$ . This algebra acts in natural way on the space  $\mathcal{H}$  of Laurent series  $\sum c_n z^n$ ; namely if  $f(z) \in \mathcal{H}$  then  $Df(z) := zf'(z)$  and  $xf(z) := -\partial_z f(z)$ .

This action can be extended to pseudodifferential operators (i.e. in the expression for the operator we can allow negative  $k$ ).

The subspace  $\mathcal{H}_+$  consisting of polynomials is invariant with respect to the action of differential operators; moreover, differential operators can be characterized as pseudodifferential operators preserving this subspace.

Let us fix an operator  $Q \in \mathcal{D}$ . We say that the elements  $u_1, \dots, u_q$  form a  $Q$ -basis of  $\mathcal{H}_+$  iff the elements  $Q^k u_i$  form a basis of  $\mathcal{H}_+$ . For example, if  $Q$  is a monic operator of order  $q$  one can take a  $Q$ -basis  $u_i = z^{i-1}$ . The  $Q$ -basis is defined up to multiplication by an invertible matrix having polynomials of  $Q$  as entries. The group of such matrices will be denoted by  $G$ .

If  $P \in \mathcal{D}$  is another differential operator we define the companion matrix  $M$  associated to the pair  $(P, Q)$  as the matrix of the coefficients in the expression

$$(1) \quad Pu_i = M_i^j(Q)u_j.$$

The entries of this matrix are polynomials with respect to  $Q$ . The matrix  $M$  depends on the choice of the  $Q$ -basis. If  $[P, Q] = \hbar$  and the basis  $u_i$  is replaced by the basis  $g_i^j(Q)u_j$  then the matrix  $M$  should be replaced by  $gMg^{-1} + \hbar \frac{dg}{dQ}g^{-1}$ . Notice that this is the standard formula for gauge transformation, hence the companion matrix can be regarded as a connection.

Using this notion we can define the quantization of a pair  $(P, Q)$  of commuting differential operators where  $Q$  is a monic differential operator. We say that the quantization leads to the pair  $(P_\hbar, Q_\hbar)$  of differential operators obeying  $[P_\hbar, Q_\hbar] = \hbar$  and having the same companion matrix in the  $Q$ -basis  $u_i = z^{i-1}$  where  $1 \leq i \leq q$ .

If the operator  $Q$  is a normalized operator of order  $q$  (i.e. the leading term is  $\partial^q$  and the subleading term vanishes) one can find such a pseudodifferential operator  $S$  of the form  $S = 1 + \sum_{k < 0} s_k(x)D^k$  that  $S^{-1}QS = D^q$ . The natural projection of the space  $V = S\mathcal{H}_+$  onto  $\mathcal{H}_+$  is an isomorphism (i.e.  $V$  belongs to the big cell of infinite-dimensional Grassmannian). Using this remark one can check that to solve the equation  $[P_\hbar, Q_\hbar] = \hbar$  we should find a space  $V_\hbar$  belonging to the big cell of Grassmannian that is invariant with respect to the multiplication by  $z^q$  and with respect to an operator  $\tilde{P}_\hbar$  having the form  $\hbar \frac{d}{dz^q} + s_\hbar(z)$  where  $s_\hbar(z)$  stands for the operator of multiplication by a Laurent series denoted by the same symbol and  $\frac{d}{dz^q}$  should be understood as  $\frac{1}{qz^{q-1}} \frac{d}{dz}$  (see Ref. [8]).

The physical quantities can be expressed in terms of the tau-function corresponding to  $V_\hbar$  or in terms of corresponding vector  $\Psi_\hbar$  in the fermionic Fock space. Notice that one can find the vector  $\Psi_\hbar$  using the following statement: if the subspace  $V$  obeys  $AV \subset V$  where  $A$  is a linear operator in  $\mathcal{H}$  having the matrix  $a_{mn}$  in the standard basis then the corresponding vector in the fermionic Fock space is an eigenvector of the operator  $\sum a_{mn} : \psi_m \psi_n^+ :$  (see Ref. [4]).

The companion matrix of the pair  $(P_\hbar, Q_\hbar)$  can be described as the matrix of  $\tilde{P}_\hbar$  in the  $z^q$ -basis  $v_i$  of  $V_\hbar$ :

$$(2) \quad \tilde{P}_\hbar v_i = M_i^j(z^q)v_j.$$

Notice that in this approach the entries of this matrix are polynomials of  $z^q$ .

The space  $V_\hbar$  has a natural  $z^q$ -basis

$$v_i = z^i + \text{lower order terms},$$

here  $0 \leq i < q$ . This basis (and the corresponding matrix  $M$ ) are defined up to triangular transformation with constant coefficients. One can say that the operators  $(P_h, Q_h)$  are obtained by means of quantization of  $(P_0, Q_0)$  if they have the same companion matrix in the natural  $z^q$ -basis of  $V_h$ . (This definition agrees with the definition in terms of  $Q$ -basis  $z^i$  of  $\mathcal{H}_+$  because the operator  $S$  transforms this basis into the natural basis of  $V_h$ .)

Sometimes it is convenient to write the equation (2) in the form

$$(3) \quad \left( \hbar \frac{d}{dz^q} + s_h(z) \right) u_i(z) = B_i^j(z) u_j$$

where  $v_i = z^i u_i$  and

$$B = (B_i^j(z)) = \left( M_i^j(z^q) z^{j-i} - \frac{i\hbar}{qz^q} \delta_i^j \right).$$

To find  $s_h(z)$  we use the condition that (3) considered as an equation for  $u_i$  should have solution with asymptotic behavior  $u_i = 1 + \dots$  where ... stands for lower order terms. It is convenient to solve at first the following auxiliary equation

$$(4) \quad \hbar \frac{d}{dz^q} w_i(z) = B_i^j(z) w_j.$$

Notice that we can introduce the covariant derivative (meromorphic connection on  $\mathbb{C}$ ) by the formula

$$\nabla = \hbar \frac{d}{dz^q} - B_i^j(z),$$

then the equation (4) specifies flat sections. Notice that the connection (4) is gauge equivalent to the connection specified by the matrix  $M$ ; we could work in terms of the latter connection.

We express  $s_h(z)$  in terms of formal diagonalization of (4). We should solve the following problem: Find a formal change of variables of the form  $w_i(z) = R_i^j(z) t_j(z)$  that diagonalizes the equation i.e. reduces it to the form

$$(5) \quad \hbar \frac{d}{dz^q} t_i(z) = \Lambda_i(z) t_i(z).$$

It is well known that such a diagonalization is possible if the leading term of the matrix  $B$  has  $q$  distinct eigenvalues. [11] Using this statement we can prove that there exist  $q$  different solutions to (3) corresponding to the coefficients  $\Lambda_i(z)$ ; namely, we can take

$$(6) \quad s_h(z) = \Lambda_i(z).$$

To prove this fact we notice first of all that after the change of variables we obtain the equation

$$(7) \quad \hbar \frac{d}{dz^q} t_i(z) = \Lambda_i^j(z) t_j(z),$$

where

$$\Lambda_i^j(z) = S_i^r B_r^m R_m^j - \hbar S_i^m \frac{dR_m^j}{dz^q}.$$

( Here  $S$  denotes the matrix inverse to the matrix  $R$ .)  
In other words,

$$(8) \quad R_k^i \Lambda_i^j(z) = B_k^m R_m^j - \hbar \frac{dR_k^j}{dz^q}.$$

We choose  $R$  in such a way that the matrix  $\Lambda$  is a diagonal matrix with entries  $\Lambda_i$ . Then it follows from (8) that for every  $r$  the series  $w_i(z) = R_i^r(z)$  satisfy the equation (3) with  $s_h(z) = \Lambda_r(z)$ .

Note that the companion matrix  $M(z^q)$  is obviously invariant with respect to transformations  $z \rightarrow \epsilon z$  where  $\epsilon^q = 1$ . It follows that the group  $C_q$  of  $q$ -th roots of 1 is a symmetry group of the equation (3). It acts on the coefficients of (5): if  $\Lambda(z)$  is one of these coefficients then  $\Lambda(\epsilon z)$  is also a coefficient.

Up to terms tending to zero as  $\hbar \rightarrow 0$  the coefficients  $\Lambda_i(z)$  coincide with the eigenvalues of the matrix  $B$ , or with the eigenvalues  $\Lambda_i^{(0)}$  of the matrix  $M$  ( up to terms of order  $\hbar$  matrices  $B$  and  $M$  are similar ).

Let us denote by  $B^{(i)}$  the  $i$ -th order term of matrix  $B$  as an expansion in  $\hbar$ . If  $R^{(0)}$  is the matrix of eigenvectors of matrix  $B^{(0)}$ , i.e.  $(R^{(0)})^{-1} B^{(0)} R^{(0)} := \Lambda^{(0)}(z)$  is a diagonal matrix with diagonal elements being eigenvalues  $\Lambda_i^{(0)}$  then

$$(9) \quad \Lambda(z) = \Lambda^{(0)}(z) + \left[ (R^{(0)})^{-1} B^{(1)} R^{(0)} - (R^{(0)})^{-1} \frac{d}{d(z^q)} R^{(0)} \right]^{\text{diag}} \hbar + o(\hbar).$$

Note that for  $\hbar = 0$  the commuting operators  $P_0, Q_0$  satisfy the algebraic equation  $A(P_0, Q_0) = 0$ , where  $A$  stands for the characteristic polynomial of the matrix  $M$ :

$$A(P_0, Q_0) = \det(P_0 I - M(Q_0)).$$

For  $\hbar \neq 0$  one can find an operator annihilating  $v_0$ . To do this we should exclude  $v_1, \dots, v_{q-1}$  from (2) or  $u_1, \dots, u_{q-1}$  from (3).

It is easier to find an operator  $\hat{A}$  annihilating  $w_0 = \rho(z)v_0$ :

$$\hat{A}w_0 = 0,$$

where  $\rho(z) = \exp(\hbar^{-1} \int s_h(z) d(z^q))$ . There is a standard way to construct  $\hat{A}$  as a differential operator with meromorphic coefficients (i.e. as a polynomial with respect to  $\partial_z$  with coefficient that are meromorphic with respect to  $z$ ).<sup>1</sup> Namely, we should consider  $w = (w_0, \dots, w_{q-1})$  as an element of  $\mathcal{F}^q$  where  $\mathcal{F}$  denotes the field of meromorphic functions. Then  $w_0 = \langle e_0, w \rangle$  where  $e_0 = (1, 0, \dots, 0)$  and  $\langle \dots \rangle$  denotes the standard bilinear inner product with values in  $\mathcal{F}$  (i.e.  $\langle a, b \rangle = \sum a_i b_i$ ). Defining  $\nabla_*$  by the formula

$$\nabla_* = \hbar \frac{d}{dz^q} + B_i^j(z),$$

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<sup>1</sup>The meromorphic connection determined by the companion matrix is flat, hence we can say that it specifies a  $D$ -module. This  $D$ -module corresponds to one differential equation with meromorphic coefficients; in the construction of  $\hat{A}$  we use this fact. This remark gives an explanation of relation between our constructions and the constructions of Ref. [1].

and using  $\nabla w = 0$  we obtain that

$$(\hbar \frac{d}{dz^q})^s w_0 = \langle \nabla_*^s e_0, w \rangle.$$

To find  $\hat{A}$  we notice that the vectors  $\nabla_*^s e_0$  with  $s = 0, \dots, q$  are linearly dependent in  $q$ -dimensional vector space over  $\mathcal{F}$ . If we can take

$$\nabla_*^q e_0 = \sum_{0 \leq s < q} a_s(z, \hbar) \nabla_*^s e_0$$

then

$$(10) \quad \hat{A} = (\hbar \frac{d}{dz^q})^q - \sum_{0 \leq s < q} a_s(z, \hbar) (\hbar \frac{d}{dz^q})^s.$$

As we have noticed another way to construct  $\hat{A}$  is to exclude  $v_1, \dots, v_{q-1}$  from (2); if this procedure leads to differential operator with meromorphic coefficients we obtain an equivalent result.

The operator  $\hat{A}$  can be regarded as quantization of the classical observable  $A$ ; compare with Ref. [3].

Let us illustrate the construction of the operator  $\hat{A}$  in terms of the matrix  $M$  in the case when  $q = 2$ . We start with the equation

$$(11a) \quad (\hbar \frac{d}{dz^2} + s_h(z))v_0 = av_0 + bv_1,$$

$$(11b) \quad (\hbar \frac{d}{dz^2} + s_h(z))v_1 = cv_0 + dv_1,$$

where  $a, b, c, d$  are the entries of matrix  $M$  (they are polynomial functions of  $z^2$ .) The function  $s_h(z)$  can be expressed in terms of the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} a & zb \\ z^{-1}c & d \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2z^2} \end{pmatrix} \hbar$$

up to terms of order  $\hbar^2$ :

$$s_h(z) = \frac{a + d \mp \sqrt{\Delta}}{2} - \left( \frac{c}{\sqrt{\Delta}} \frac{d}{d(z^2)} \left( \frac{\sqrt{\Delta} \mp (a - d)}{2c} \right) + \frac{1}{2z^2} \right) \hbar + o(\hbar),$$

where  $\Delta = (a - d)^2 + 4bc$ .

The commuting differential operators  $P_0, Q_0$  with companion matrix  $M$  obey the characteristic equation of this matrix :  $A(P_0, Q_0) = 0$ , where

$$A(x, y) = x^2 - (a + d)x + ad - bc.$$

This means that

$$s_h(x) = x(y) + O(\hbar)$$

where  $x(y)$  stands for one of branches of the expression for  $x$  obtained from the equation  $A(x, y) = 0$ .

Excluding  $v_1$  from (11) we obtain that

$$\hat{A}(\rho(z)v_0) = 0,$$

where

$$(12) \quad \hat{A} = (\hat{x}^2 - (a + d)\hat{x} + (ad - bc)) - \hbar b^{-1} (b'\hat{x} - (b'a - ba')),$$

$\rho(z) = \exp(\hbar^{-1} \int s_{\hbar}(z) d(z^2))$ ,  $\hat{x} = \hbar \frac{d}{d(z^2)}$ ,  $\hat{y} = z^2$ ,  $a, b, c, d$  are polynomials in  $\hat{y}$ .

The calculation based on the formula (10) leads to the same expression.

We took as a starting point for construction of quantum curve a pair of commuting differential operators  $P_0, Q_0$ . Such pair of operators obeys an algebraic equation, therefore they can be considered as meromorphic functions on algebraic curve. Let us show that one can start with a pair of meromorphic functions  $u, v$  on the algebraic curve  $C$  of genus  $g$  in the construction of quantum curve. This is reminiscent of Eynard-Orantin topological recursion [2], but our conditions on functions  $u, v$  are different.

For simplicity let us consider the case when the functions  $u, v$  have only one pole located at non-singular point  $c \in C$ . Let us consider a subspace  $W$  of the space of meromorphic functions on  $C$ , that is invariant with respect to multiplication by  $u$  and  $v$  and has a basis  $s_n$  such that  $s_n$  has a pole of order  $n$  at the point  $c$ .<sup>2</sup>

The multiplication by  $u$  and  $v$  specifies commuting operators on  $W$ . The natural identification of  $W$  and  $\mathcal{H}_+$  allows us to consider these operators as commuting differential operators. We can use these operators as an input in the construction of quantum curve. However, this is not necessary: one can construct the quantum curve directly from  $u$  and  $v$ . Namely, we should take a  $v$ -basis  $f_1, \dots, f_q$  in  $W$ . (This means that the functions  $v^k f_i$  form a basis in  $W$ . The number  $q$  is equal to the order of the pole of  $v$ .) Using this basis we can construct a matrix of the operator of multiplication by  $u$ :

$$u f_i = M_i^j(v) f_j.$$

We use this matrix to construct a quantum curve, i.e. a pair of differential operators with the commutator equal to  $\hbar$  having the matrix as the companion matrix.

Until now we have considered scalar differential operators. However, with small modifications we can apply our considerations to matrix differential operators. In this case we should replace the space  $\mathcal{H}$  by the space  $\mathcal{H} \otimes \mathbb{C}^r$  of vector-valued Laurent polynomials. The entries of the companion matrix associated become  $r \times r$  matrices. The function  $v$  should have  $r$  poles.

## EXAMPLES

In the examples below we start with functions  $u, v$  defined on the algebraic curve specified by the equation  $A(u, v) = 0$ .

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<sup>2</sup>If we try to use as  $W$  the space of meromorphic functions having a pole only at  $c$  we discover that there are  $g$  gaps (Weierstrass gaps) in the sequence of  $n$ 's serving as the order of poles. Therefore we should weaken our conditions allowing poles elsewhere. One can allow poles at non-special divisor of degree  $g - 1$ .

1.  $u^2 - v = 0$ . Functions  $u$  and  $v$  can be parametrized as meromorphic functions  $u = z$  and  $v = z^2$  on  $\mathbb{P}^1$ . Then the space  $W = \mathcal{H}_+$  is the space of all meromorphic functions only have poles at  $\infty$ . The  $v$ -basis for  $W$  is given by  $f_0 = 1$  and  $f_1 = z$ . Thus we can construct the matrix  $(M_i^j(z))$  of the operator of  $u$ -multiplication on the basis  $\{f_i\}$

$$M = \begin{pmatrix} 0 & 1 \\ z^2 & 0 \end{pmatrix}.$$

Then the operator  $\hat{A}$  can be constructed according (12).

$$\hat{A}(v_0) = (\hat{u}^2 - \hat{v})(v_0) = 0,$$

where  $\hat{u} = \hbar \frac{d}{d(z^2)} \pm z - \frac{\hbar}{4z^2}$ ,  $\hat{v} = z^2$ .

2.  $u^3 - v^2 = 0$ . Functions  $u$  and  $v$  can be parametrized as  $u = z^2$  and  $v = z^3$ . The space  $W = \mathcal{H}_+$  is the same as Example 1. The  $v$ -basis for  $W$  is given by  $f_0 = 1$ ,  $f_1 = z$  and  $f_2 = z^2$ . Thus the matrix of  $u$ -multiplication on this  $v$ -basis would be

$$M = \begin{pmatrix} 0 & 0 & 1 \\ z^3 & 0 & 0 \\ 0 & z^3 & 0 \end{pmatrix}.$$

The operator  $\hat{A}$  can be constructed according to (10) by finding coefficients  $a_s(z, \hbar)$  of  $\nabla_*^3(e_0)$  in terms of  $\nabla_*^i(e_0)$  ( $0 \leq i < 3$ ), where  $\nabla_*$  is defined as  $\hbar \frac{d}{d(z^q)} + M_i^j(z)$ .

$$\hat{A}(v_0) = (\hat{u}^3 - \hat{v}^{-1} \hbar \hat{u}^2 - \hat{v}^2)(v_0) = 0,$$

where  $\hat{u} = \hbar \frac{d}{d(z^3)} + s_\hbar(z)$ ,  $\hat{v} = z^3$ . One can take  $s_\hbar(z) = z^2 - \frac{\hbar}{3z^3}$ . Other possibilities are  $s_\hbar(\epsilon z) = \epsilon^2 z^2 - \frac{\hbar}{3z^3}$  and  $s_\hbar(\epsilon^2 z) = \epsilon z^2 - \frac{\hbar}{3z^3}$  where  $\epsilon = e^{2\pi i/3}$ .

Another way to construct the operator  $\hat{A}$  is to exclude  $v_1, v_2$  from equations (2):

$$\tilde{P}_\hbar \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ z^3 & 0 & 0 \\ 0 & z^3 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ z^3 v_0 \\ z^3 v_1 \end{pmatrix}.$$

We obtain

$$(\hat{v}^{-1} \hat{u})(\hat{v}^{-1} \hat{u}) \hat{u} v_0 = v_0,$$

where  $\hat{u} = \tilde{P}_\hbar$ ,  $\hat{v} = z^3$ . Two results are equivalent. (They differ by a factor  $\hat{v}^2$ .)

**3. General case  $u^q - v^p = 0$  where  $p$  and  $q$  are coprime.** Functions  $u$  and  $v$  are parametrized as  $u = z^p$ ,  $v = z^q$ . The space  $W = \mathcal{H}_+$  is the same as Example 1 and 2. The  $v$ -basis is given by  $\{z^i\}_{0 \leq i < q}$ . The matrix  $M$  of  $u$ -multiplication on the  $v$ -basis is

$$M = (M_i^j) = \left( v^{\frac{p+i-\sigma(i)}{q}} \delta_{\sigma(i)}^j \right), \quad (0 \leq i, j < q),$$

where  $\sigma$  denotes the permutation on  $q$ -indices  $\{0, 1, \dots, q-1\}$  induced by shifting each index by  $p$ , in the sense of module  $q$ , namely  $\sigma(i) \equiv p + i \pmod{q}$ . Then the matrix  $M$  can be considered as the companion matrix of operator  $\tilde{P}_\hbar$  in the  $z^q$ -basis  $v_i$  according to

equation (2). The differential operator  $\tilde{P}_h$  is written by  $\hbar \frac{d}{dz^q} + s_h(z)$ . To find  $s_h(z)$ , it is convenient to use equation (3). In this case

$$B = (B_i^j(z)) = \left( z^p \delta_{\sigma(i)}^j - \hbar \frac{i}{qz^q} \delta_i^j \right) := A_0 z^p - \hbar A_1 z^{-q},$$

where  $A_0 = (\delta_{\sigma(i)}^j)$ ,  $A_1 = (\frac{i}{q} \delta_i^j)$ . Then  $s_h(z)$  is given by (6), where  $\Lambda(z)$  is defined in (9).  $R^{(0)}$  is the matrix of eigenvectors of  $A_0 z^p$ . Since  $A_0$  is the matrix of permutation  $\sigma$ , it is easy to see  $R^{(0)}$  can be described by Vandermonde matrix  $(x_j^i)$ , where  $x_j = \epsilon^j$ ,  $\epsilon = e^{2\pi i/q}$ . And the diagonal matrix  $\Lambda^{(0)}(z) = (z\epsilon^i)^p \delta_i^j$  is consisted of corresponding eigenvalues. If we denote the inverse matrix of  $R^{(0)}$  by  $P = (P_j^i)$ , then the matrix multiplication  $PR^{(0)} = I$  can be considered as the evaluation of polynomials  $P^i(x) := \sum_{l=0}^{q-1} P_l^i x^l$  at  $x_j$ , such that  $P^i(x_j) = \delta_j^i$ . Therefore  $P^i(x)$  can be calculated by means of Lagrange interpolation formula:

$$P^i(x) = \frac{\prod_{k \neq i} (x - x_k)}{\prod_{k \neq i} (x_i - x_k)} = \frac{x^q - 1}{qx_i^{q-1}(x - x_i)}.$$

Then the  $i$ -th diagonal elements of the conjugation  $-PA_1 z^{-q} R^{(0)}$  can be calculated

$$-\frac{1}{qz^q} \sum_k k P_k^i x_i^k = \frac{x_i}{qz^q} \frac{d}{dx} \Big|_{x=x_i} P^i(x) = -\frac{x_i^2}{q^2 z^q} \lim_{x \rightarrow x_i} \frac{(q-1)x^q - qx^{q-1}x_i + 1}{(x - x_i)^2} = \frac{1-q}{2qz^q}.$$

And in (9)  $\frac{d}{d(z^q)} R^{(0)}$  vanishes because  $R^{(0)}$  does not dependent on  $z$ . Thus we obtained  $q$  values of  $s_h$  corresponding to eigenvalues of  $A_0$ : for the eigenvalue 1 we get  $s_h(z) = z^p + \frac{1-q}{2qz^q} \hbar + o(\hbar)$ , and for eigenvalues  $\epsilon^i$  we get  $s_h = (\epsilon^i z)^p + \frac{1-q}{2qz^q} \hbar + o(\hbar)$ ,  $(0 < i < q)$ .

Let us prove that the higher order ( $o(\hbar)$ ) terms vanish. One of possible ways to give a proof is to check that the equations (2) or (3) have solutions that satisfy the necessary conditions.<sup>3</sup>

We will prove that for  $s_h(z) = z^p + \frac{1-q}{2qz^q} \hbar$  there exists a solution  $u = (u_i)$  for (3) having the desired asymptotic behavior:  $u_i = 1 + \dots$ , where  $\dots$  means lower order terms in  $z$ .

Let's denote  $\alpha = \frac{1-q}{2q}$ , so that  $s_h(z) = z^p + \hbar \alpha z^{-q}$ . If we express  $u$  as a Laurent series  $u = (u_i) = \sum_{j \geq 0} \xi_j z^{-j}$  with  $q$ -dimensional vectors  $\xi_i$  as coefficients, then according to (3) we have  $\xi_0 = (1, \dots, 1)^T$  and  $\xi_j$  should satisfy the following formula:

$$(13) \quad (A_0 - 1)\xi_i = 0 \quad (0 \leq i < p+q),$$

$$(14) \quad (A_0 - 1)\xi_i = \hbar \left( A_1 + \alpha - \frac{i-p-q}{q} \right) \xi_{i-p-q} \quad (i \geq p+q).$$

We can show that (13) and (14) determine  $\xi_i$  completely and uniquely.

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<sup>3</sup>We should prove that there exists a desired solution for every eigenvalue of matrix  $A_0$ , however, it is sufficient to consider one of this eigenvalues; we will give the proof for the eigenvalue 1. (The group  $C_q$  of  $q$ -th roots of unity is a symmetry group of the equation (3); it acts transitively on the eigenvalues of  $A_0$ .)



As we noticed 1 is one of  $q$  different eigenvalues of  $A_0$ . Let's denote its one-dimensional eigenspace by  $V_1$  (it is spanned by  $\xi_0$ ). Then for  $i = 1, \dots, p+q-1$ , we have  $\xi_i \in V_1$ . They can be determined later by using (14).

It is also useful to note that image space of linear transformation  $A_0 - 1$  has codimension 1, its standard orthogonal complement space is  $V_1$ . Therefore we have an orthogonal decomposition  $\mathbb{R}^n = \text{Im}(A_0 - 1) \oplus V_1$ .

Then for  $i = p+q$ , we can verify that  $\hbar(A_1 + \alpha)\xi_0 \in \text{Im}(A_0 - 1)$ , because it is orthogonal to  $\xi_0$  because of the value of  $\alpha$ .<sup>4</sup> Then (14) makes sense and  $\xi_{p+q}$  can be solved up to an element in  $V_1$ . Let's write  $\xi_{p+q} = \tilde{\xi}_{p+q} + c_{p+q}\xi_0$ , according to the decomposition  $\mathbb{R}^n = \text{Im}(A_0 - 1) \oplus V_1$ . Then  $\tilde{\xi}_{p+q}$  is completely determined. The scalar  $c_{p+q}$  can be determined later.

For  $i = p+q+s$ , ( $0 < s < p+q$ ), (14) can be written as

$$(A_0 - 1)\xi_{p+q+s} - \hbar(A_1 + \alpha)\xi_s = -\hbar\frac{s}{q}\xi_s.$$

Since  $\xi_s \in V_1$ , we can check the l.h.s. is in  $\text{Im}(A_0 - 1)$  too.<sup>5</sup> But the r.h.s. is in  $V_1$ . Therefore both sides must be 0, which implies  $\xi_s = 0$ . Then  $\xi_{p+q+s} \in V_1$ .

For  $i = 2(p+q)$ , one can use the decomposition  $\xi_{p+q} = \tilde{\xi}_{p+q} + c_{p+q}\xi_0$ , take the inner product  $\xi_0$  to (14) and get

$$0 = \hbar\xi_0^T(A_1 + \alpha - \frac{p+q}{q})(\tilde{\xi}_{p+q} + c_{p+q}\xi_0).$$

Then  $c_{p+q} = \xi_0^T(\frac{A_1 + \alpha}{p+q} - \frac{1}{q})\tilde{\xi}_{p+q}$  is determined. And it shows that the r.h.s. of (14) is in  $\text{Im}(A_0 - 1)$ , therefore  $\xi_{2(p+q)}$  can be determined up to an element in  $V_1$ , namely  $\xi_{2(p+q)} = \tilde{\xi}_{2(p+q)} + c_{2(p+q)}\xi_0$ , where the first component is determined and  $c_{2(p+q)}$  can be determined by looking at the case when  $i = 3(p+q)$ .

By using this procedure and induction, we can prove that  $\xi_i = 0$  if  $i \neq k(p+q)$ , and  $\xi_{k(p+q)}$  can be uniquely determined. Hence the solution  $u = \sum_j \xi_j z^{-j}$  is determined uniquely.

The operator  $\hat{A}$  can be derived from (10) or from (2). Let's use (2).

Let  $\hat{u}$  be the operator  $\tilde{P}_h$ , and  $\hat{v}$  be the operator of  $z^q$ -multiplication. Then equation (2) can be formulated as

$$\hat{u}(v_i) = M_i^j(\hat{v})(v_j) = \hat{v}^{\frac{p+i-\sigma(i)}{q}}(v_{\sigma(i)}).$$

Therefore

$$\hat{v}^{-\frac{p+i-\sigma(i)}{q}}\hat{u}(v_i) = v_{\sigma(i)}.$$

Then the action of the operator  $\hat{v}^{-\frac{p+i-\sigma(i)}{q}}\hat{u}$  is simply the permutation of indices on  $v_i$ . Hence by repeating the action  $q$ -times, the permutation circles back to identity, then we

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<sup>4</sup>Note that the value of  $\alpha$  is crucial here. In this value changes, then  $\hbar(A_1 + \alpha)\xi_0 \notin \text{Im}(A_0 - 1)$ , the recursion formula would not hold.

<sup>5</sup>Again the value of  $\alpha$  is important for the same reason.

get

$$\left(\hat{v}^{-\frac{p+\sigma^{q-1}(0)-\sigma^q(0)}{q}}\hat{u}\right)\left(\hat{v}^{-\frac{p+\sigma^{q-2}(0)-\sigma^{q-1}(0)}{q}}\hat{u}\right)\cdots\left(\hat{v}^{-\frac{p+0-\sigma(0)}{q}}\hat{u}\right)(v_0)=v_0.$$

To avoid the use of negative power of  $\hat{v}$ , we multiply  $\hat{v}^{p+q-1}$  on both sides. Then operator  $\hat{A}$  is obtained:

$$\hat{A}(v_0) = [\hat{v}^{p+q-1}(\hat{v}^{-m_q}\hat{u})\cdots(\hat{v}^{-m_1}\hat{u}) - \hat{v}^{p+q-1}](v_0) = 0$$

where  $m_i = \frac{p+\sigma^{i-1}(0)-\sigma^i(0)}{q}$ .

We have described the quantum curve obtained by quantization of the classical curve  $u^q - v^p = 0$ . We have proven that the subspace  $V_h$  (the point of Grassmannian corresponding to this quantum curve) is invariant with respect to the operator

$$\tilde{P}_h = \hbar \frac{d}{dz^q} + z^p + \frac{1-q}{2qz^q}\hbar.$$

This means that the space  $V_h$  coincides with the point of Grassmannian corresponding to the  $(p, q)$ -minimal model coupled to 2D gravity. ( See Ref. [4], [6] and [9].)

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